Cosmological spacetimes not covered by a constant mean curvature slicing

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Abstract

We show that there exist maximal globally hyperbolic solutions of the Einstein-dust equations which admit a constant mean curvature Cauchy surface, but are not covered by a constant mean curvature foliation.

1. Introduction

The most widely advocated candidate to be used as a choice of time in cosmological spacetimes, i.e., spacetimes which are

a) globally hyperbolic with a compact Cauchy surface, and

b) solutions of Einstein's equations coupled to 'reasonable'(see Sect. 2) non-gravitational fields

is that which is specified by a constant mean curvature ('CMC') foliation. Supporting this advocacy is the belief that a generic cosmological spacetime admits a global CMC foliation which is unique up to leaf-preserving diffeomorphism. While this belief may turn out to be true, it was shown some years ago [1] that there are some cosmological spacetimes which do not admit any CMC Cauchy surfaces. Here we show further that there are also cosmological spacetimes which admit CMC Cauchy surfaces but do *not* admit a global *foliation* by such hypersurfaces.

The spacetimes which we show have this property are quite special *: the matter fields in them are pressureless fluids, i.e. dust, which are contrived to form 'shell-crossing singularities', and the spacetime fields are all invariant under a T^2-isometry group. While the latter feature is merely a convenience, it may well be that the formation of a shell-crossing singularity is crucial to the failure of the CMC slicing to cover the spacetime. This issue needs further study. In any case, we believe it is useful to know that there are cosmological spacetimes in which CMC-based time exists for a while, but does not cover the whole globally hyperbolic spacetime.

Our results here, to an extent, build on previous work by one of us[12]. In that work, it was shown that there are cosmological spacetimes (with T^2 isometry and satisfying the Einstein-dust equations) which, starting from initial data with finite constant mean curvature τ_0, develop a (shell-crossing) singularity arbitrarily soon - $\tau_0 + \epsilon$ - in CMC time, provided the solution exists that long. What we show here is that, while the singularity stops the CMC foliation from continuing past $\tau_0 + \epsilon$, for some (generic?) choices of initial data, the spacetime continues to develop globally hyperbolically in regions which are spatially distant from the singularity. Hence, there are spacetime regions which are left uncovered by the CMC slicing. Note that while examples of spacetimes with incomplete CMC foliations can be constructed trivially by cutting pieces out of certain spacetimes (see, for example, p. 472 of [13]), such examples are not maximal globally hyperbolic developments of initial data, as are our examples described here.

To prove the existence of spacetimes with these properties, we first (Sect. 2) briefly discuss cosmological spacetimes satisfying the Einstein-dust equations, noting especially results concerned with stability, and we describe the special form of the fields and some of the field equations if we assume that the spacetimes are T^2 symmetric. Next, in Sect. 3, we describe the special sets of initial data which were used in [12] to produce singularities in arbitrarily short CMC time. Then in Sect. 4, we specify the 'quiet zone' restrictions on the data which we use to produce our present results, and we state and prove the results in Theorem 3 in that section.

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* Note that, thus far, the spacetimes which are known not to admit any CMC hypersurfaces are also special in that they are solutions of the Einstein-dust field equations. No vacuum solutions are known to have this property

2. Cosmological spacetimes with dust

As noted earlier, a spacetime $(M^4, g_{\alpha\beta}, \Psi)$ with metric $g_{\alpha\beta}$ and non-gravitational fields Ψ is called *cosmological* if (M^4, g) is globally hyperbolic with compact Cauchy surface, and if $g_{\alpha\beta}$ and Ψ together satisfy the coupled Einstein-matter system:

$$G_{\alpha\beta}[g] = 8\pi T_{\alpha\beta}[g, \Psi] \tag{1a}$$

$$\mathcal{F}[g, \Psi] = 0 \tag{1b}$$

where $T_{\alpha\beta}[g, \Psi]$ is the stress-energy tensor of Ψ and $g_{\alpha\beta}$, and where $\mathcal{F}[g, \Psi] = 0$ represents the field equations which the chosen field theory imposes on $g_{\alpha\beta}$ and Ψ. The globally hyperbolic condition implies that $M^4 = \Sigma^3 \times \mathbf{R}$ for some compact three-dimensional manifold Σ^3, and also implies that any spacelike embedding of Σ^3 in (M^4, g) is a Cauchy surface[2]. Global hyperbolicity does not, however, guarantee that the spacetime $(M^4, g_{\alpha\beta}, \Psi)$ is uniquely determined by initial data on a Cauchy surface. For this property to hold, the PDE system (1) must be *hyperbolic* in an appropriate sense[6]. This PDE hyperbolicity condition, along with an energy condition (weak, strong, or dominant)[7]) on the energy-momentum tensor, is what we mean by 'reasonable' non-gravitational fields in the definition of cosmological spacetimes.

Our focus in this paper is the Einstein-dust field theory, in which the non-gravitational fields consist of a non-negative function μ (the proper energy density of the matter) and a unit vector field u^α (the four-velocity of the matter). The stress-energy tensor for this theory takes the form

$$T_{\alpha\beta} = \mu u_\alpha u_\beta, \tag{2}$$

and the non-gravitational field equations are just those derived from the Bianchi identity

$$\begin{aligned} 0 &= \nabla_\alpha G^\alpha{}_\beta[g] \\ &= \nabla_\alpha T^\alpha{}_\beta[g, \mu, u] \end{aligned} \tag{3}$$

One readily verifies from (2) that the weak, strong and dominant energy conditions are all satisfied by the Einstein-dust theory. In contrast, it is *not* so easy to verify that the PDE system for this theory is hyperbolic. The procedure of reduction to a symmetric hyperbolic system, which straightforwardly shows hyperbolicity for the Einstein equations coupled to many matter fields, runs into difficulties in the case of dust. (In contrast it does work for the Einstein-perfect fluid equations with a reasonable equation of state and strictly positive pressure.) By rewriting (1)-(3) in the equivalent form

$$u^\gamma \nabla_\gamma G_{\alpha\beta} = -\mu(\nabla_\gamma u^\gamma) u_\alpha u_\beta \tag{4a}$$

$$u^\gamma \nabla_\gamma u_\beta = 0 \tag{4b}$$

$$\nabla_\gamma(\mu u^\gamma) = 0 \tag{4c}$$

Choquet-Bruhat is able to show that the Einstein-dust field equations are hyperbolic in the Leray sense[3],[9].

It is important for our present purposes to know not only that one can determine the spacetime $(M^4, g_{\alpha\beta}, \mu, u^\alpha)$ from knowledge of initial data on a Cauchy surface $\Sigma^3_{\tau_0}$, but also

that at least for a finite (in time) spacetime neighbourhood $M^4_{(\tau_0-\lambda,\tau_0+\lambda)}$ of $\Sigma^3_{\tau_0}$, the fields in this determined spacetime $(M^4_{(\tau_0-\lambda,\tau_0+\lambda)}, g_{\alpha\beta}, \mu, u^\alpha)$ are *stable* under perturbations of the initial data. The theorem which guarantees this stability for PDE systems which are hyperbolic in the sense of Leray was proven by Choquet-Bruhat in [4] using harmonic coordinates. Applied to the Einstein-dust equations (4) this result says the following.

Proposition 1. (Choquet-Bruhat [4]) Let $(M^4_0, g^0_{\alpha\beta}, \mu_0, u^\alpha_0)$ be a cosmological spacetime satisfying the harmonically reduced Einstein-dust equations and let $(\hat{M}^4_0, \hat{g}^0_{\alpha\beta}, \hat{\mu}_0, \hat{u}^\alpha_0)$ be the restriction of this solution to a region $\hat{M}^4_0$ bounded between a pair of Cauchy surfaces in M^4. Let $(\hat{g}^0_{\alpha\beta}, \hat{\mu}_0, \hat{u}^\alpha_0)$ be contained in the Sobolev space $H_7(\hat{M}^4)$, and for a fixed Cauchy surface Σ^3 in $\hat{M}^4$, let the corresponding Cauchy data C_0 be contained in $H_7(\Sigma^3)$. There exists an open set $W \subset H_7(\Sigma^3)$ with $C_0 \in W$ such that every set of initial data $C \in W$ generates a cosmological spacetime $(\hat{M}^4, \hat{g}_{\alpha\beta}, \hat{\mu}, \hat{u}^\alpha)$ which satisfies the harmonically reduced Einstein-dust equations, and also $(\hat{g}_{\alpha\beta}, \hat{\mu}, \hat{u}^\alpha) \in H_7(\hat{M}^4)$. Moreover the mapping which takes a point of W to the corresponding solution is continuous (in fact differentiable) with respect to the H_7 topology.

The spacetimes that we show here have an incomplete CMC slicing are Einstein-dust cosmological spacetimes, and in addition have $\Sigma^3 = T^3$ and admit a T^2 isometry group which acts spatially. In a neighbourhood of some fixed CMC initial slice Σ^3_0, the metric for such a spacetime can be written in the form[11]

$$g = -\alpha^2 dt^2 + A^2[(dx + \beta^1 dt)^2 + a^2 \tilde{g}_{AB}(dy^A + \beta^A dt)(dy^B + \beta^B dt)] \tag{5}$$

Here we use coordinates (x, y^2, y^3, t), where x and y^A are spatial periodic coordinates on T^3, with $\partial/\partial y^A$ being Killing fields, and t is a time coordinate (possibly, but not necessarily CMC). The functions α, A, β^1 and β^A are all functions of x and t (periodic in x), as is the two-dimensional unit determinant Riemannian metric $\tilde{g}_{AB}$; the function a depends on t only. We also require that the density function μ depend only on x and t, and that the matter velocity field u^α take the form $u = vA^{-1}\partial/\partial x + w\partial/\partial t$, with v a function of x and t only and with w determined by the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$.

Substituting the metric (5) into the Einstein-matter equations (1a), (2) and (3), and using the assumptions just stated, one obtains an explicit formulation of the Einstein-dust Cauchy problem for a class of T^2-symmetric cosmologies in terms of α, A, β^1, β^A, a, μ and v. While we will need to use the local existence and stability results for the full Cauchy problem - see Proposition 1 above - we shall only be working with a couple of the equations in explicit form. The two we will need - they are constraints on the choice of initial data - are [12]:

$$\partial^2/\partial x^2(A^{1/2}) = -\tfrac{1}{8}A^{5/2}[\tfrac{3}{2}(K - \tfrac{1}{3}\tau)^2 - \tfrac{2}{3}\tau^2 + \sigma^2 + 16\pi\mu(1+v^2)] \tag{6a}$$

$$\partial K/\partial x + 3A^{-1}(\partial A/\partial x)K - \tfrac{1}{3}A^{-3}\partial/\partial x(A^3\tau) - \lambda = 8\pi A\mu(1+v^2)^{1/2}v, \tag{6b}$$

where τ is the mean curvature of the chosen initial data (generally a function of α, β^1, β^A, a and their time derivatives), where K is an eigenvalue of the second fundamental form of

the initial data - specifically one has:

$$K = \tfrac{2}{3}\alpha^{-1}[a^{-1}\partial a/\partial t + \partial\beta^1/\partial x] + \tfrac{1}{3}\tau \tag{7}$$

and where σ and λ are certain functions of the initial data which we need not specify, since they vanish for the cases we consider below.

3. Initial Data for Spacetimes with Finite CMC Slicing

In our examples the mechanism which is envisaged as stopping the CMC slicing from proceeding is the formation of 'shell-crossing singularities'. It is appropriate at this point to discuss this feature of dust spacetimes in some more detail. The general idea is that self-gravitating dust particles have a tendency to collide with each other, creating singularities. More specifically, in a spacetime with an isometry group with two dimensional orbits, a shell of dust particles which are related to each other by the symmetry moves in a coherent way. If two of these shells collide then the intermediate shells are trapped between them, so that the matter density is forced to blow up. Let us give a formal definition in the case of T^2 symmetry which is of interest here.

Definition. Consider a T^2-symmetric cosmological spacetime which is a solution of the Einstein-dust equations on which is defined a time coordinate t with compact level surfaces whose range is (t_-, t_+). Suppose that there is a time t_0 with $t_- < t_0 < t_+$ and two distinct dust particles which move orthogonal to the group orbits with the property that the distance between their world lines, as measured in the hypersurfaces t=const. tends to zero as $t \to t_+$. Then we say that the spacetime develops a shell-crossing singularity as $t \to t_+$.

In the examples which we discuss in the following we will not prove that a shell-crossing singularity develops. However the strategy of the proof is guided by the idea that that is what is happening. To construct these examples, we rely on the initial value formulation, so our aim is to find simple initial data sets for Einstein-dust cosmological spacetimes, which are apt to lead to shell-crossing in the future, and which of course satisfy the Einstein constraint equations. We do this using the conformal method [5], as adapted to the T^2 symmetric metric (5) and the corresponding matter variables discussed in Sect. 2. As in [5], we proceed as follows:

– Choose (for some fixed constant $t_0 < 0$)
(i) $\tau = t_0$
(ii) $\tilde{g}_{ab} = \delta_{ab}$ (so $\lambda = 0$)
(iii) $a = 1$
(iv) second fundamental form so that $\sigma = 0$
(v) $\tilde{\mu}(x)$ any smooth positive function on S^1 with

$$\tilde{\mu}(\pi - x) = \tilde{\mu}(x) \tag{8a}$$

and
(vi) $\tilde{v}(x)$ any smooth function on S^1 with

$$\tilde{v}(\pi - x) = -\tilde{v}(x) \tag{8b}$$

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– Determine $\tilde{K}(x)$ by solving

$$\partial\tilde{K}/\partial x = 8\pi\tilde{\mu}(1+\tilde{v}^2)\tilde{v} \tag{9a}$$

and $A(x)$ by solving

$$\partial^2/\partial x^2(A^{1/2}) = -\tfrac{1}{8}A^{5/2}[\tfrac{3}{2}A^{-3}(\tilde{K}-\tfrac{1}{3}t_0)^2 - \tfrac{2}{3}t_0^2 + 16\pi A^{-4}\tilde{\mu}(1+\tilde{v}^2)] \tag{9b}$$

– Set

$$\mu = A^{-4}\tilde{\mu} \tag{10a}$$
$$v = \tilde{v} \tag{10b}$$
$$K = \tfrac{1}{3}t_0 + A^{-1}(\tilde{K}-\tfrac{1}{3}t_0) \tag{10c}$$

For a given choice of t_0, $\tilde{\mu}(x)$ and $\tilde{v}(x)$, this method produces a solution of the constraints so long as equations (9a) and (9b) admit a solution $\tilde{K}$ and A. The reflection symmetry conditions (8a)-(8b) guarantee that the right hand side of (9a) satisfies

$$\int_{S^1}\tilde{\mu}(1+\tilde{v}^2)\tilde{v}dx = 0,$$

from which it follows that (9a) admits a solution (unique up to a constant). Sub and supersolution techniques [8] show that (9b) admits a solution as well, with that solution bounded below and above by the subsolution A_- and the supersolution A_+ respectively:

$$A_- = t_0^{-1}(24\pi B)^{1/2} \tag{11a}$$
$$A_+ = \mathrm{Max}\{t_0^{-1}(48\pi\|\tilde{\mu}\|_\infty(1+\|\tilde{v}\|_\infty^2)^{1/2}, t_0^{-2/3}(\tfrac{9}{2}\|\tilde{K}-\tfrac{1}{3}t_0\|_\infty^2)^{2/3}\} \tag{11b}$$

Here $B := \mathrm{Min}\ \tilde{\mu} > 0$.

Note that, a priori, β^1 and α may be chosen freely. If one wishes to maintain a CMC foliation into the future and past, however, then α must be chosen to satisfy

$$\partial^2\alpha/\partial x^2 + A^{-1}(\partial A/\partial x)(\partial\alpha/\partial x) = \alpha A^2[\tfrac{3}{2}(K-\tfrac{1}{3}t_0)^2 + \tfrac{1}{3}t_0^2 + 4\pi\mu(1+v^2)] - A^2 \tag{12}$$

Similarly, to maintain a fixed x coordinate range, say $[0, 2\pi]$, one must restrict β^1. We shall presume that this has been done.

So far, we have not encoded shell-crossing into the initial data. This is readily done: one chooses a pair of points x_1 and x_2 in the interval $(0,\pi)$, and one chooses the function v so that $v(x_1) = 1$ and $v(x_2) = -1$. These data describe a pair of dust particles (among others) starting at x_1 and x_2 at t_0, with initial spatial velocities $+1$ and -1. If one uses CMC time - $\tau(t) = t$ - then the equations of motion for these particles are

$$dx/dt = \alpha A^{-1}v(1+v^2)^{-1/2} - \beta^1 \tag{13a}$$
$$dv/dt = -A^{-1}\partial\alpha/\partial x(1+v^2)^{-1/2} + \alpha K v \tag{13b}$$

From these equations, as well as from the Einstein-dust equations, one obtains upper and lower bounds for changes in the velocities of the particles in finite CMC time [12]. Since these bounds are independent of $|x_2 - x_1|$, one expects that for sufficiently small $|x_2 - x_1|$, shell crossings are inevitable.

Now, as a consequence of local existence theorems for the Einstein-dust PDE system, we know that for every choice of sufficiently smooth initial data, a cosmological spacetime consistent with those data - i.e., a cosmological *development* of that data - exists for at least a finite proper time into the future and into the past of the initial Cauchy surface at t_0. In the special case of T^2 -symmetric spacetimes, local existence theorems in terms of CMC coordinates have been proved [11]; so, one knows in addition that there is, in every maximal development of initial data constructed as above according to conditions (i)-(vi) and equations (9)-(10), a finite CMC time slicing in a neighbourhood of the initial surface.

While one expects shell-crossing to stop a cosmological development from proceeding in proper time, it is possible a priori that CMC time could proceed to its limits - which are 0 and $-\infty$ in a cosmological spacetime with $\Sigma^3 = T^3$ - in such a spacetime. As shown in [12], this is not the case. Indeed one has

Proposition 2. [12] Let $t_0 < 0$ and $\epsilon > 0$. There exist sets of CMC ($\tau = t_0$) initial data for the Einstein-dust system such that in any development of the data, the CMC slicing which exists in a neighbourhood of the initial surface cannot be extended past either $t_0 + \epsilon$ or $t_0 - \epsilon$.

The basic idea of the proof is to use the Einstein-dust equations - including the CMC equation (12) - to obtain controls on the geometric quantities α, A, and K in a finite CMC time interval about t_0, and then argue that for $|x_2 - x_1|$ small enough, the dust particles starting at x_1 and x_2 (with initial velocities $+1$ and -1 as described above, and with metric described by equation (13)) must intersect within CMC time $t_0 + \epsilon$, if the solution exists that long. An intersection of this type would violate regularity and so CMC time must stop. Similarly, if one chooses $v(x)$ with $v(x_1) = -1$ and $v(x_2) = +1$, the intersection occurs in the past, within $t_0 - \epsilon$. For details, see [12].

We note two facts regarding this result which are important for our work in the next section. First, we note that shell-crossing is essentially a local phenomenon. Specifically, it follows from the details of the proof of Proposition 2 that regardless of what is happening outside of the interval $[x_1, x_2]$, one can choose data inside $[x_1, x_2]$ which results in shell-crossing occurring arbitrarily soon. Second, it is a trivial consequence of Proposition 2 as stated that given any sequence of positive numbers T_m converging to zero, one can choose a sequence of Einstein-dust initial data C_m such that in any spacetime development of C_m, CMC slicing extends no further (in CMC time) than $t_0 + T_m$.

4. Spacetimes Extending Past the CMC Slicing

We now consider sets of initial data of the form described above, with shell-crossing built in, but with a 'quiet region' $[x_-, x_+]$ far away from $[x_1, x_2]$. In this quiet region, we choose

$$\tilde{\mu}(x) = 1, \quad x_- < x < x_+ \tag{14a}$$

$$\tilde{v}(x) = 0, \quad x_- < x < x_+ \tag{14b}$$

It follows from (9a) that $(\partial\tilde{K}/\partial x)(x) = 0$ for $x_- < x < x_+$, so making a convenient choice of constant, we have $\tilde{K}(x) = \frac{1}{3}t_0$ on that interval. Thus we find that, modulo the conformal factor A^2, which solves (9b) on S^1, we have in $[x_-, x_+]$ initial data for a spatially flat Friedmann-Robertson-Walker cosmology.

Our claim is that for all sets of initial data of this type, regardless of how soon shell-crossing occurs, A is sufficiently well-behaved in $[x_-, x_+]$ that in the maximal spacetime development of a given set of such data, the domain of dependence of $[x_-, x_+]$ extends a finite proper time into the future and into the past, and hence extends past the CMC slicing, if that is cut off sufficiently quickly. This is the content of our main result:

Theorem 3. There exist Einstein-dust cosmological spacetimes, maximal developments of Cauchy data, which contain a CMC Cauchy surface but cannot be foliated by a CMC slicing.

Proof. We fix an initial time $t_0 \in (-\infty, 0)$, and on the circle we fix a pair of disjoint intervals* $[x_1, x_2]$ (the shell-crossing region) and $[x_-, x_+]$ (the quiet region). We restrict attention to sets of data which satisfy conditions (i)-(vi) in the above procedure for constructing initial data and equations (9)-(10), but with conditions (8a) and (8b) on $\tilde{\mu}(x)$ and $\tilde{v}(x)$ replaced by the conditions that:

a) $\tilde{\mu}(x)$ is a positive smooth function on S^1 with

$$\tilde{\mu}(x) = 1, \quad x_- < x < x_+ \tag{15a}$$

and

$$\tilde{\mu}(\pi - x) = \tilde{\mu}(x) \tag{15b}$$

b) $\tilde{v}(x)$ is any smooth function on S^1 with

$$|\tilde{v}(x)| \leq 1 \tag{16a}$$

$$\tilde{v}(x) = 0, \quad x_- < x < x_+ \tag{16b}$$

$$\tilde{v}(\pi - x) = -\tilde{v}(x) \tag{16c}$$

$$\tilde{v}(x_1) = 1 \tag{16d}$$

and

$$\tilde{v}(\hat{x}) = -1 \tag{16e}$$

for some $\hat{x} \in (x_1, x_2)$. As discussed above, we can solve (9a) for $\tilde{K}$, and we obtain that $\tilde{K} = \frac{1}{3}t_0$ on $[x_-, x_+]$.

Now in accord with the comments at the end of Sect. 3, it follows from the details of the proof of Proposition 2 in [12] that one can choose a sequence of initial data C_m of the form just described - with $\tilde{\mu}(x)$ the same for all m - such that the CMC slicing in any

* Without loss of generality, we assume that $[x_1, x_2] \cap [\pi - x_+, \pi - x_-] = \phi$ as well as $[x_1, x_2] \cap [x_-, x_+] = \phi$

development of C_m extends no further than $t_0 \pm T_m$. What we now wish to show is that A and its derivatives restricted to $[x_-, x_+]$ are controlled uniformly, for all sets of data in this sequence C_m.

First, we recall the sub and super solutions (11) for A on S^1. The sub solution is clearly the same for all sets C_m. For the super solution A_+, while $\|\tilde{\mu}\|_\infty$ and $\|\tilde{v}\|_\infty$ are independent of m, the quantity $\|\tilde{K} - \frac{1}{3}t_0\|_\infty$ may not be. However, it follows from (9a) that

$$\begin{aligned} |\tilde{K} - \tfrac{1}{3}t_0| &= 8\pi \left| \int_{S^1} \tilde{\mu}(1+\tilde{v}^2)\tilde{v} \right| \\ &\leq 16\pi^2 \|\tilde{\mu}\|_\infty (1 + \|\tilde{v}\|_\infty^2) \|\tilde{v}\|_\infty^2 \end{aligned} \tag{17}$$

which is independent of m. Hence we have upper and lower positive bounds for $A(x)$ for the entire sequence. Next, as detailed in [12], we use a generalization of an estimate of Malec and Ó Murchadha [10] to argue that $\partial A/\partial x$ is also uniformly bounded along the sequence.

We now note the form of the Lichnerowicz equation (9b), which expresses $\partial^2 A/\partial x^2$ in terms of A, $\tilde{\mu}$, $\tilde{K}$, $\tilde{v}$ and τ. We know from above that A is uniformly bounded, and we have chosen $\tilde{\mu}$ and τ to be fixed (and bounded) independent of the sequence. The only functions that change as m grows, and could cause trouble for $\partial^2 A/\partial x^2$ are $\tilde{v}$ and $\tilde{K}$. However, in $[x_-, x_+]$ we have $\tilde{v} = 0$ and $\tilde{K} - \frac{1}{3}t_0$ for all m; it follows that *in* $[x_-, x_+]$ we have $\partial^2 A/\partial x^2$ uniformly bounded along the sequence. Taking spatial derivatives of the Lichnerowicz equation (9b), we may similarly argue that spatial derivatives of A of all orders are, at least in $[x_-, x_+]$, uniformly bounded.

With these controls on A, we have established uniform C^∞ bounds on the sequence C_m of initial data, restricted to $[x_-, x_+]$. We may now apply the Cauchy stability statement of Proposition 1, from which we infer the following: there exists a pair of finite positive numbers λ and μ (with $\lambda \leq \frac{1}{2}(x_+ + x_-)$) and there exists a fixed harmonic coordinate system (x, y^1, y^2, t) on the spacetime region

$$\hat{M}^4 = (x_- + \lambda, x_+ - \lambda) \times S^1 \times S^1 \times (t_0 - \nu, t_0 + \nu)$$

such that for each set of initial data C_m, there is a spacetime $(\hat{M}^4, g_m)$ with the properties
(i) $(\hat{M}^4, g_m)$ is a solution of the Einstein-dust equations
(ii) the metric g_m induces the initial data C_m on the hypersurface $t = t_0$
(iii) the metric coefficients of g_m and g_m^{-1} written in terms of the coordinates (x, y^1, y^2, t) are all C^∞ uniformly bounded

In other words, because of the 'quietness' built into each set of initial data C_m on the region $[x_-, x_+]$, the maximal spacetime development g_m of each C_m lasts at least for a common (harmonic) time ν. Note that we use harmonic coordinates here because the Cauchy stability proposition is proven [4] using harmonic coordinates.

Let us now fix a point $(x_*, y_*^A) \in (x_- + \lambda, x_+ + \lambda) \times S^1 \times S^1$; for each metric g_m, we consider the timelike geodesic γ_m which is orthogonal to the initial surface

$$(x_- + \lambda, x_+ + \lambda) \times S^1 \times S^1 \times \{t_0\}$$

at the point (x_*, y_*^A, t_0). Since g_m and its inverse are both well-behaved in harmonic coordinates, one readily determines that for each γ_m, there is a positive number L_m which

measures the proper time length of γ_m from the point (x_*, y_*^A, t_0) on the initial surface to the point where γ_m leaves $\hat{M}^4$. Moreover, the uniform boundedness of g_m and g_m^{-1} implies the existence of some $L > 0$ such that $L_m > L$ for all m.

By construction, in any spacetime development (M^4, g_m) of the initial data C_m, there is a CMC slicing in the neighbourhood of the initial surface, with the slicing lasting no longer than T_m in CMC time and with $\lim_{m\to\infty} T_m = 0$. If we consider the length $\mathcal{L}_m$ of γ_m restricted to the CMC region, we obtain the formula

$$\mathcal{L}_m = \int_{t_0}^{t_0+T_*} \alpha(\gamma(t))dt \tag{18}$$

where t is CMC time.

From the CMC equation (12) for α (which holds for all CMC $t \in [t_0, t_0 + T_m]$) we note that at any point where α achieves a maximum, we have

$$0 \geq A^2\alpha[\tfrac{3}{2}(K - \tfrac{1}{3}t)^2 + \tfrac{1}{3}t^2 + 4\pi\mu(1 + v^2)] - A^2 \tag{19}$$

which implies (recalling that $t < 0$) that

$$\alpha \leq 3/t_0^2 \tag{20}$$

Hence from (18), we have

$$\mathcal{L}_m \leq (3/t_0^2)T_m \tag{21}$$

But $\lim_{m\to\infty} T_m = 0$, so that $\lim_{m\to\infty} \mathcal{L}_m = 0$. This tells us that for some sufficiently large value of m, we have in $\hat{M}^4$ (and in any development $\bar{M}^4$ containing $\hat{M}^4$) $\mathcal{L}_m < L_m$. Thus in $\bar{M}^4$, the geodesic γ_m leaves the region of CMC slices before it passes out of the region with harmonic coordinates.

Note that the arguments of this section do not rely essentially on the fact that the matter model is dust; a perfect fluid with pressure would do just as well. The only point where the fact that the matter model is dust is used is in applying Proposition 2. If an analogue of Proposition 2 could be proved for a perfect fluid with pressure then a generalization of Theorem 3 to that matter model would follow. It seems plausible that an analogue of Proposition 2 does hold for a fluid with pressure, due to the formation of shock waves, but the proof would be very different from that in the case of dust. On the other hand, there are other forms of matter for which the analogue of Proposition 2 is false; this has been proved for collisionless matter and wave maps in [11]

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